Costruction of classic exact solutions for Tricomi equation

Gianluca Argentini
Research & Development Dept., Riello Burners - Italy
gianluca.argentini@rielloburners.com
gianluca.argentini@gmail.com

2010, October

Abstract
A formula to construct classic exact solutions to Tricomi partial differential equation. The steps to obtain this formula require only elementary resolution of a simple system of first order PDEs.

Keywords
second order PDE, fluid dynamics, system of first order partial differential equations.

The (generalized) Tricomi equation is the second order partial differential equation

$$\partial_{xx} f + a(x)\partial_{yy} f = 0 \tag{1}$$

where $f(x, y)$ and $a(x)$ are regular functions. In the case $a(x) = x$, (1) is the classic Tricomi equation, as described by Prof. Francesco in his work dated 1923 [5]. The equation is an abstraction of the Euler equation on a 2D fluid motion in the case of a flow speed near the sonic condition. For Tricomi equation some exact solutions formulas exist ([4] or [2]), and solutions to some particular boundary values problems are known [1]. Also, some results about weak solutions are found [3].

Let be $u = \partial_x f = f_x$, $v = \partial_y f = f_y$. Then Tricomi equation can be written as a system of two first order equations:

$$\begin{cases} u_x(x, y) + a(x)v_y(x, y) = 0 \\ u_y(x, y) - v_x(x, y) = 0 \end{cases} \tag{2}$$

We define, for convenience reasons, a function $t = t(x, y) = v(x, y)$. From the first equation we can write

$$u(x, y) = -\int_a^x [a(s)\ t_y(s, y)]\, ds + g(y) \tag{3}$$

where $a \in \mathbb{R}$ and g is an arbitrary function of real variable. Now we have to find the condition that g and t must satisfy to verify the second equation of the system (2). This equation is now, from the usual rule of derivation of integrals depending on parameters,

$$-\int_a^x [a(s)\ t_{yy}(s,y)]\, ds + g'(y) = t_x \tag{4}$$

and hence

$$g(y) = \int_b^y \left[t_x(x,r) + \int_a^x [a(s)\ t_{yy}(s,r)]\, ds \right] dr \tag{5}$$

with $b \in \mathbb{R}$. But g must depend only on y: using the Fundamental Theorem of the Integral Calculus we have

$$0 = \partial_x g(y) = \int_b^y [t_{xx}(x,r) + a(x)\ t_{yy}(x,r)]\, dr \qquad \forall y \tag{6}$$

If t is a solution of Tricomi equation (1), previous condition is verified. Note that, integrating with respect to y in $g(y)$ formula, the expression for u becomes

$$u(x,y) = \int_b^y t_x(x,r) dr - \int_a^x [a(s)\ t_y(s,b)]\, ds \tag{7}$$

So, we have found that if t is a solution of Tricomi equation, then a function f such that $f_x = u$ and $f_y = t$, with u given by (7), is a solution too. Hence, from $f_y = t$ we can write

$$f(x,y) = \int_b^y t(x,r) dr + h(x) \tag{8}$$

where h is an arbitrary function of real variable. From $f_x = u$ follows

$$\int_b^y t_x(x,r) dr + h'(x) = \int_b^y t_x(x,r) dr - \int_a^x [a(s)\ t_y(s,b)]\, ds \tag{9}$$

so that

$$h(x) = -\int_a^x \int_a^q [a(s)\ t_y(s,b)]\, ds\ dq \tag{10}$$

We have so proven that *if $t = t(x,y)$ is a generic solution of Tricomi equation, then the following formula gives a solution of the same equation too*:

$$f(x,y) = \int_b^y t(x,r) dr - \int_a^x \int_a^q [a(s)\ t_y(s,b)]\, ds\ dq \tag{11}$$

For *a-posteriori* verification, let be t such that $t_{xx} + a(x) t_{yy} = 0$. We have

$$
\begin{aligned}
f_{xx} &= \int_b^y t_{xx}(x,r)dr - a(x)\ t_y(x,b) = \\
&= -\int_b^y a(x)t_{yy}(x,r)dr - a(x)\ t_y(x,b) = \\
&= -a(x)\ t_y(x,y) + a(x)\ t_y(x,b) - a(x)\ t_y(x,b) = \\
&= -a(x)\ t_y(x,y) = \\
&= -a(x) f_{yy}
\end{aligned} \tag{12}
$$

so the solving formula (11) is verified.

Example 1. Let be $f_{xx} + x f_{yy} = 0$ the original Tricomi equation. Note that $t(x,y) = y$ is a (trivial) solution. Then, from (11) with $a = b = 0$, the function

$$
f(x,y) = \int_0^y y\ dr - \int_0^x \int_0^q s\ ds\ dq = \frac{1}{2}y^2 - \frac{1}{6}x^3 \tag{13}
$$

is a solution too.

Example 2. Let be $f_{xx} + \cos(x) f_{yy} = 0$ a generalized Tricomi equation. Note that $t(x,y) = y$ is a (trivial) solution. Then, from (11) with $a = b = 0$, the function

$$
f(x,y) = \int_0^y y\ dr - \int_0^x \int_0^q \cos(s)\ ds\ dq = -1 + \frac{1}{2}y^2 + \cos(x) \tag{14}
$$

is a solution too.

Note that formula (11) can be used as a *solutions machine*: starting from the most simple not null solution, that is from $t(x,y) = 1$, it can be used for iterative construction of solutions.

Also, formula (11) can be used to find solutions to some particular boundary values problems for Tricomi equation. For example, in the case $a(x) = x$ and $b = 0$, if the function $t(x,y)$ is such that $t_y(x,0) = 0\ \forall x$ (condition satisfied e.g. by $t = -\frac{1}{6}x^3 + \frac{1}{2}y^2$), then from (11) follows

$$
f(x,y) = \int_0^y t(x,r)dr \tag{15}
$$

so that $f(x,0) = 0\ \forall x \in \mathbb{R}$. Hence, if $t_y(x,0) = 0\ \forall x$, then $f(x,y) = \int_0^y t(x,r)dr$ *is a solution of the boundary values differential problem*

$$
\partial_{xx} f + a(x)\partial_{yy} f = 0\ , \qquad f(x,0) = 0 \quad x \in \mathbb{R} \tag{16}
$$

If t is a solution of (1) satisfying the boundary conditions $t(x,0) = g(x)$, with g a function of real variable, then from (11) follows that $f_y(x,0) = g(x)$, that is *f is a solution of the problem with Cauchy-Neumann condition*

$$\partial_{xx} f + a(x)\partial_{yy} f = 0 \ , \qquad \partial_y f(x,0) = g(x) \quad x \in \mathbb{R} \tag{17}$$

References

[1] M. Cinquini Cibrario, Equazioni a derivate parziali di tipo misto, *Milan Journal of Mathematics*, vol. 25, **1**, Springer, 1955

[2] http://eqworld.ipmnet.ru/en/solutions/lpde/lpde402.pdf, 2010

[3] K. Yagdjian, Global Existence for the n-Dimensional Semilinear Tricomi-Type Equations, *Comm. in partial differential equations*, Vol. 31, **6**, 2006

[4] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, 2002

[5] F. Tricomi, Sulle equazioni differenziali alle derivate parziali di secondo ordine di tipo misto, *Memorie della Reale Accademia dei Lincei*, S. 5^a, Vol. XIV, fasc. 7, 1923